Dynamics of associative memory with a general transfer function

Ioan Opriş*
Department of Physics, University of Bucharest, Bucharest-Măgurele, Romania
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The Glauber dynamics of magnetic systems has been extended to the case of neural networks with a general odd response function. We have derived a set of recursion relations for the overlap parameter, noise average, and noise variance taken as macrovariables of the process describing the dynamics of associative memory. The retrieval process has been studied then for a hyperbolic tangent transfer function by the self-consistent signal to noise ratio method. The fatigue effect of the real neuron has been taken into account. The phase diagrams of the retrieval process reveal an enhanced storage capacity for a certain set of values of the parameters. Finally, a set of equations for the overlap parameters in the case of continuous asynchronous dynamics with nonmonotone neurons has been analytically derived.

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I. INTRODUCTION

The neural network models of the associative memory are dynamical systems with associated attractors to the cognitive events. A very well known example is the Hopfield model [1,2] successfully carried out by Amit *et al.* [3,4] with the equilibrium statistical mechanics tools. The dynamics of a neural network with a general response function [5] is much more difficult to treat than equilibrium properties because there is no general framework corresponding to the Boltzmann-Gibbs equilibrium theory. Even the stochastic master equation of Glauber dynamics has been considered only for a monotonic transfer function of the hyperbolic tangent type [6–10]. Despite these difficulties, the approximative treatment of the retrieval process performed by Amari and Maginu [11] gave satisfaction for various transfer functions [10,5,12,13], the only macrovariables used being the overlap of the current state onto an embedded pattern and the variance.

The aim of this paper is to develop a scheme to treat the dynamics of both synchronous and asynchronous associative memories with a general odd transfer function including nonmonotonic cases. To carry out this task we have employed the method of Horn and Usher [14] by using the discrete time master equation to describe the time evolution of the network state. The macrovariables of the process are the overlap, the variance given through the signal to noise ratio analysis [11], and the noise average. We deal here with a self-consistent extraction process of the signal from noise which finally yields an enhanced storage capacity. This mechanism of enhancing the storage capacity is different from those involving the pseudoinverse method [15] or the partial reversed method [16].

The paper is organized as follows. In Sec. II we develop the general framework of the associative memory with a general transfer function [5] and successively derive the generalized macrovariables recursion relations together with the time-dependent probability and the discrete master equation. In Sec. III we study the retrieval process of an associative memory having a simple hyperbolic tangent output function by following a self-consistent signal to noise ratio method. We extend the Glauber dynamics method to a general odd transfer function including nonmonotonic functions in Sec. IV. Finally, the conclusions and suggestions on improvement are discussed in Sec. V.

II. ASSOCIATIVE MEMORY DYNAMICS

The dynamics of the neural network describes the change of variables in time. Let us consider a neural network of N two-state neurons $S_i = \pm 1$ $(i = 1, ..., N)$, which interact through the couplings J_{ij} given by the Hebb rule $J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu$. The input-output function f sets the relationship between the neuron's new state $S_i(t+1)$ and the previous network state $\{S(t)\}$

$$S_i(t+1) = f\left(\sum_{j=1}^{N} J_{ij} S_j(t)\right) \qquad (J_{ii} = 0). \tag{1}$$

Morita *et al.* [16-18] have introduced the odd nonmonotonic function $f(h)$, which can be written as a product between the ordinary sigmoid output function and a function $g_{\text{inh}}(h,a)$ as

$$f(h) = \tanh(\beta h) g_{\text{inh}}(h,a), \tag{2}$$

the output inverting function being

$$g_{\text{inh}}(h,a) = \tanh[-\gamma(\mid h \mid - a)/2] \tag{3}$$

with γ a positive constant and a the threshold which makes a transfer function nonmonotone. In the limit $\beta \to \infty$, $\tanh \beta h \to \text{sgn(h)}$ and for $\gamma \to \infty$, $g_{\text{inh}}(h,a)$ resembles a truncated sombrero shape playing the role of inhibition for an input value greater than a.

In order to implement any general input-output rela-

*Electronic address: opris@ifa.ro

tion into the present discrete-time discrete-state network including the nonmonotonic ones, it is thus natural to introduce the process

$$\mathrm{Prob}[S_i(t+1)] = \frac{1}{2}[1 + S_i(t+1) f(h_i(t))], \qquad (4)$$

with a general function f. The absolute value of f should not exceed 1 because otherwise the probabilistic interpretation (4) does not make sense.

The postsynaptic potential (PSP) or the local field $h_i(t)$ of the ith neuron is expressed as $h_i(t) = \sum_j J_{ij} S_i(t) = \sum_\mu \xi_i^\mu m^\mu(t) = \xi_i^1 m^1(t) + N_i(t)$, where the overlap $m^\mu(t)$ with the μth pattern is defined by $m^\mu(t) = \frac{1}{N}\sum_i \xi_i^\mu S_i(t)$. Applying the signal to noise ratio method we can split the PSP into two terms, the pure signal and the noise $h_i(t) = \xi_i^1 m^1(t) + N_i(t)$ with the noise term defined by $N_i(t) = \frac{1}{N}\sum_j \sum_{\mu\neq 1} \xi_i^\mu \xi_j^\mu S_j(t)$.

We further self-consistently decompose [19] the noise part into a pure noise component $N_i^0(t)$ and an output proportional term $\alpha S_i(t)$, the decomposition relation being $N_i(t) = N_i^0(t) + \alpha S_i(t)$, with α the memory loading rate. To include the fatigue effect given by the threshold contribution [14] one takes the following recursion relation for the noise term: $N_i(t+1) = \lambda N_i(t) + \alpha S_i(t+1)$ with $\lambda < 1$. Using the notation $m = m^1$, the overlap parameter becomes $m(t+1) = \frac{1}{N}\sum_i \xi_i^1 f(\xi_i^1 m(t) + N_i(t))$, through multiplication by ξ_i^1 and statistical averaging.

Because the function f is an odd function in input, the factor $\xi_i^1 = \pm 1$ can be moved into the argument for a class of input-output functions. [For example, in the case of $f(x) = \tanh(x) g(x)$ we get $\xi f(x) = \tanh(\xi x) g(x)$, $g(x)$ being an even function.] The new expression of $m(t+1)$ reads

$$m(t+1) = \frac{1}{N}\sum_i f(m(t) + \xi_i^1 N_i(t)). \qquad (5)$$

Let us denote the value $\xi_i^1 N_i(t)$ at location i by z and assume that this value is distributed with the probability $P(z,t)$ given by [14]

$$P(z,t) = \frac{1}{N}\sum_i \delta(\xi_i^1 N_i(t) - z); \qquad (6)$$

this probability allows us to write $m(t+1)$ as an integral equation

$$m(t+1) = \int dz P(z,t) f(m(t) + z). \qquad (7)$$

The value of z at location i changes in one iteration to $\lambda z \pm 1$ with the probabilities

$$\pi^+(m(t),z) = \frac{1}{2}[1 + f(m(t)+z)],$$

$$\pi^-(m(t),z) = \frac{1}{2}[1 - f(m(t)+z)] \qquad (8)$$

extended to the general odd input-output function. These relations give us the discrete master equation as a recursion relation

$$P(z,t+1) = \frac{1}{\lambda}\left[\pi^+\left(m(t), \frac{z-\alpha}{\lambda}\right) P\left(\frac{z-\alpha}{\lambda}, t\right) + \pi^-\left(m(t), \frac{z+\alpha}{\lambda}\right) P\left(\frac{z+\alpha}{\lambda}, t\right)\right]. \qquad (9)$$

Since the most important features of the probability distribution are the average ζ and the standard deviation, let us replace $P(z,t)$ by an expression which contains these values and can be easily manipulated,

$$P(z,t) = \frac{1}{2}\delta(z - \zeta(t) - \sigma(t)) + \frac{1}{2}\delta(z - \zeta(t) + \sigma(t)). \qquad (10)$$

Replacing the master equation by its first two moments, i.e., expectation values of z and z^2, we are led to the following set of recursion relations

$$\begin{aligned} m(t+1) &= \frac{1}{2} f(m(t) - \zeta(t) - \sigma(t)) \\ &\quad + \frac{1}{2} f(m(t) - \zeta(t) + \sigma(t)), \\ \zeta(t+1) &= \lambda\zeta(t) + \alpha m(t+1), \\ \sigma^2(t+1) &= \lambda^2\sigma(t)^2 + \lambda\sigma(t)[f(m(t) - \zeta(t) - \sigma(t)) \\ &\quad + f(m(t) - \zeta(t) + \sigma(t))] \\ &\quad + 1 - m^2(t+1). \end{aligned} \qquad (11)$$

These expressions are different from those obtained in the Amari-Maginu framework [11].

III. ANALYSIS OF THE RETRIEVAL PROCESS

The analysis of the retrieval process is carried out for a hyperbolic tangent transfer function because the mechanism responsible for the enhancement of storage capacity is not caused by the nonmonotonic function as it was expected. Here we deal with a self-consistent extraction of the signal from the noise in a recurrent manner.

In Fig. 1 we have plotted the solution of Eqs. (11) (the $m - \zeta - \sigma$ set) for $T = 0.015$, $\lambda = 0.1$, $\alpha = 0.5$ and initial conditions $m = 0.22$ and $\zeta = \sigma = 0.1$. The retrieval process exhibits for small values of ζ more or less the same behavior as that obtained by Amari and Maginu [11]. The difference consists in that our model is biologically motivated by the fatigue effect and by the dynamical threshold incorporated in the noise recursion relation.

A more convincing argument that our model works as an associative memory would be the attraction basin of a memory state which comes from Fig. 2. By plotting the time development of the overlap parameter for $T = 0.015$, $\lambda = 0.1$, $\alpha = 0.5$, and initial overlap values between $m = 0$ and $m = 1$, with increasing ratio 0.05 one can observe the convergence of solutions to a fixed-point attractor for $m > m_c$, m_c being the critical initial overlap which gives the boundary of the basin of attraction. For $m < m_c$ the retrieval process fails. The other initial conditions are the same as in Fig. 1 ($\zeta = \sigma = 0.1$).

Taking the solutions of the system (11) for $T = 0.05$

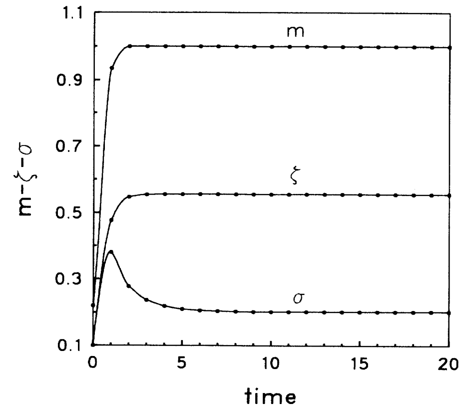

FIG. 1. Graphic solution of Eqs. (11) (the $m-\zeta-\sigma$ set) for $T=0.015$, $\lambda=0.1$, $\alpha=0.5$, and initial conditions $m=0.22$ and $\zeta=\sigma=0.1$.

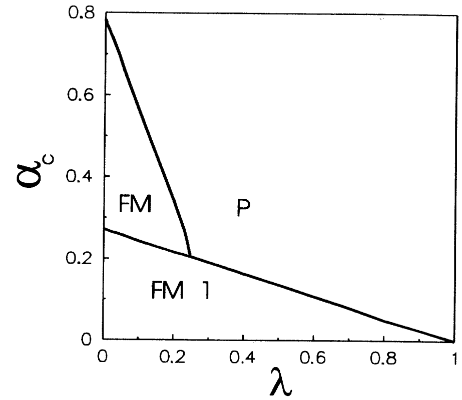

FIG. 3. Retrieval process phase boundary showing the critical storage capacity versus fatigue parameter λ. The parameters are $T=0.05$ with the same initial conditions for the $m-\zeta-\sigma$ set. At $\lambda=0$ the storage capacity is $\alpha_c=0.793$. The pure FM solutions located under the lower curve, denoted by FM1, have the final overlap $m=1$ and a first-order phase transition to the others FM solutions having final overlap $m<1$. The upper line gives a second-order phase transition between FM solutions and the paramagnetic ($m=0$) P phase.

with the same initial conditions for the $m-\zeta-\sigma$ set as those in Fig. 2, we have plotted in Fig. 3 the retrieval process phase boundary showing the critical storage capacity α_c versus the fatigue parameter λ. The diagram gives a clear-cut separation between the paramagnetic phase in which $m\to 0$ and the ferromagnetic one below the separation curve, corresponding to successful retrieval. The ability of the network to recall an enhanced number of patterns is obtained when the fatigue vanishes $\lambda=0$, the storage capacity being $\alpha_c=0.793$. Increasing λ, the storage capacity α_c decreases to zero in the $\lambda\to 1$ limit. The pure FM solutions are located under the lower curve and have the final overlap $m=1$. The pure FM phase denoted by FM1 is separated through a first-order phase transition line by the others FM solutions having the final overlap $m<1$. The upper boundary line corresponds to a second-order phase transition between FM solutions and the paramagnetic P phase.

In Fig. 4 the phase diagram $T=f(\alpha_c)$ of the retrieval process is plotted, showing a family of curves for $\lambda=0$, 0.1, 0.15, 0.2, and 0.25. Each curve gives the separation boundary between the FM phase and the paramagnetic one. Increasing the λ parameter to 1, the boundary approaches 0 and the area of the FM phase in α_c-T coordinates practically vanishes. Thus the effect of fatigue causes the reduction of associative memory performances, as expected.

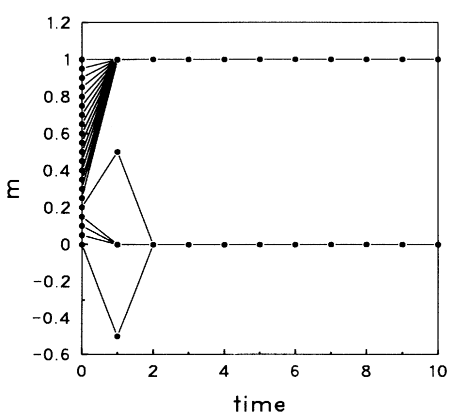

FIG. 2. Time development of the overlap parameter for $T=0.015$, $\lambda=0.1$, $\alpha=0.5$, and initial overlap values between $m=0$ and $m=1$ with increasing ratio 0.05. The other initial conditions are the same as in Fig. 1 ($\zeta=\sigma=0.1$).

IV. CONTINUOUS TIME DYNAMICS

This step considers the continuous time dynamics of the neural network with nonmonotone neurons. The dynamics is given by the equation

$$\tau\frac{du_i(t)}{dt}=-u_i(t)+\sum_j J_{ij}f(u_j(t)). \quad (12)$$

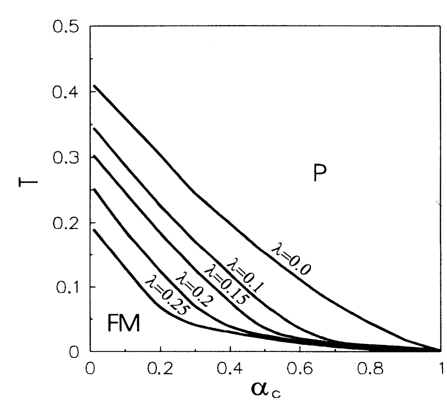

FIG. 4. Phase diagram $T=f(\alpha_c)$ of the retrieval process showing a family of curves for $\lambda=0$, 0.1, 0.15, 0.2, and 0.25.

Here $u_i(t)$ denotes the instantaneous potential of the ith neuron, f the output function, and J_{ij} the Hebbian couplings. The asynchronous updating relation is

$$S_i(t + \Delta t) = f(h_i(t)), \tag{13}$$

with the probability

$$\text{Prob}[S_i(t + \Delta t)] = \frac{1}{2}[1 + S_i(t)f(h_i(t))], \tag{14}$$

where we have taken the S_i variables as soft spins [6] with continuous values between $+1$ and -1, f the nonmonotonic function given by Eq. (2), and $h_i(t)$ the effective local field.

Incorporating the nonmonotonic input-output function into the transition rates, we consider the Glauber dynamics governing the time evolution of the microscopic state $\{S_i\}$ of the neural network. The transition rates are

$$w(S_i \to -S_i) = \frac{1}{2}[1 - S_i f(h_i\{S\})],$$
$$w(-S_i \to S_i) = \frac{1}{2}[1 + S_i f(h_i\{S\})]. \tag{15}$$

The Glauber dynamics of the network is then given [8,9] by the master equation

$$\frac{\partial}{\partial t}P(S_1, ..., S_N; t) = -\sum_i w(S_i \to -S_i)P(S_1, ..., S_i, ..., S_N; t) + \sum_i w(-S_i \to S_i)P(S_1, ..., -S_i, ..., S_N; t). \tag{16}$$

Since p patterns are embedded as memories in the network through the connections, one can divide the system of N neurons into 2^p sublattices $I(x)$ defined [20,21] by $I(x) = \{i; \xi_i = x\}, x \in \{-1, 1\}^p$, with the sublattice magnetization $m(x,t) = \frac{1}{|I(x)|}\sum_{i\in I(x)} S_i(t)$.

In the thermodynamic limit $N \to \infty$, one defines the probability $p(t \mid x)$ to have a $+1$ spin at time t in the x sublattice. The master equation for one spin reads

$$\frac{d}{dt}p(t \mid x) = -p(t \mid x) + \frac{1}{2}[1 + f(h\{x\})]. \tag{17}$$

Since the overlaps $m^\mu(t)$ are expressed through the average $m^\mu(t) = \langle x_\mu m(x,t)\rangle$, finally, we obtain a set of p coupled equations for the overlaps $m^\mu(t)$ given by

$$\frac{d}{dt}m^\mu(t) = -m^\mu(t) + \left\langle x_\mu f\left(\sum_\mu x_\mu m^\mu(t)\right)\right\rangle. \tag{18}$$

These relations are very general; they are useful for a wide spectrum of aspects in neural network research.

V. CONCLUSION

In conclusion, this paper develops the idea of extending the Glauber dynamics from magnetic systems to the case of neural networks with general odd response functions. The set of recursion relations of Horn and Usher [14] was extended to the macroscopic variables describing the dynamics of the associative memory retrieval process in the self-consistent signal to noise ratio framework. We have solved Eqs. (11) for a hyperbolic tangent transfer function and plotted the phase diagrams showing the boundary between the FM and paramagnetic phases. Our phase diagrams of the retrieval process reveal an enhanced storage capacity of $\alpha \to 1$ when temperature $T \to 0$ and the fatigue vanishes. Finally, a continuous time evolution set of overlap equations for nonmonotone neurons was analytically derived. This extension helps the investigation of any neural network having an odd output function and interesting phenomena such as temporal association and correlated data induced transitions [22]. The biological relevance of nonmonotonic firing rate was pointed out by Horikawa [23]. Monte Carlo simulations and a more detailed analysis are planned for a forthcoming paper.

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[1] J.J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79,** 2554 (1984).
[2] J.J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **81**, 3088 (1984).
[3] D.J. Amit, H. Gutfreund, and H. Sompolinsky, Phys. Rev. A. **32**, 1007 (1985).
[4] D.J. Amit, H. Gutfreund, and H. Sompolinsky, Ann. Phys. (N.Y.) **173**, 30 (1987).
[5] H. Nishimori and I. Opriş, Neural Networks **6**, 1061 (1993).
[6] R. Kühn and S. Bös (private communication).
[7] H. Rieger, M. Schreckenberg, and J. Zittartz , Z. Phys. B **72**, 523 (1982).
[8] M. Shiino, J. Stat. Phys. **59**, 1051 (1990).
[9] M. Shiino, H. Nishimori, and M. Ono, J. Phys. Soc. Jpn. **58**, 763 (1989).
[10] H. Nishimori and T. Ozeki, J. Phys. A **22,** 859 (1993).
[11] S. Amari and K. Maginu, Neural Networks **1**, 63 (1988).
[12] H. Nishimori and I. Opriş, in *Proceedings of the 1993 IEEE International Conference on Neural Networks* (IEEE, San Francisco, 1993), pp. 353–358.
[13] H. Nishimori, T. Ozeki, and I. Opriş, *Computer Aided Innovation of New Materials II* (North-Holland, Amsterdam, 1993), pp. 383–388.
[14] D. Horn and M. Usher, Phys. Rev. A **40,** 1036 (1989).
[15] T. Kohonen, IEEE Trans. Comput. **C-23**, 59 (1974).
[16] M. Morita, Neural Networks **6**, 115 (1993).
[17] M. Morita, S. Yoshizawa, and K. Nakano (private com-

munication).
[18] S. Yoshizawa, M. Morita, and S. Amari, Neural Networks **6**, 167 (1993).
[19] M. Shiino and T. Fukai (private communication).
[20] J.L. van Hemmen and R. Kühn, Phys. Rev. Lett. **57**, 913 (1986).
[21] W. Gerstner and J.L. van Hemmen, Biol. Cybern. **67**, 195 (1992).
[22] Y. Nakamura and T. Munakata, Phys. Rev. E **49**, 1775 (1994).
[23] Y. Horikawa, in *Proceedings of the 1993 IEEE International Conference on Neural Networks* (IEEE, San Francisco, 1993), pp. 473–478.